Phase synchronization and nonlinearity decision in the network of chaotic flows

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The dynamics of a network of globally coupled chaotic flows is reduced to that of a single chaotic flow as the result of the phase synchronization. The mechanism of the nonlinearity decision among the flows is clarified and a simple decision rule is presented which holds in almost the entire range of the couplings and in a wide class of nonlinear flows. The key observation is that final attractors represent the ''motion of the center of mass'' of the network. The nonlinearity of the final attractors can be controlled by couplings. [S1063-651X(97)04203-7]

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I. INTRODUCTION

The brain is a network of a huge number of chaotic neurons and it is supposed that the synchronization over cells of the network is requisite for intelligence and pattern recognition (e.g., [1,2]). In a recent paper [3] we reported an amazing phenomenon called *phase synchronization*. We observed it in a simple network of globally coupled N nonlinear flows in which the parameters of N_1 flows are set in the chaotic regime, and others ($N_2 = N - N_1$) in the periodic regime. Starting from random initial values, the like N_1 and N_2 flows first synchronize among themselves and form two clusters, one chaotic and the other periodic, each with the original nonlinearity, and then the two clusters metamorphose into two final attractors, with some mutually decided nonlinearity. The final attractors are perfectly synchronizing each other in the phase and their orbits are precisely similar but different in the size and in positions in the phase space.

Thus the phase synchronization we found concerns flows of distinct nonlinearities and consists of two parts; the formation of new metamorphosed attractors and the precise phase locking among final attractors. Rosenblum, Pikovsky and Kurths [4] independently found a similar phenomenon where two or many flows with the same nonlinearity synchronize in phase even if they are given by different angular velocities with the difference of some few tens of percent. We should note that difference in the angular velocities ($\Delta\omega$) only amounts to the difference in the linear contributions ($\dot{x}_i = -\omega_i y_i + \cdots, \dot{y}_i = \omega x_i + \cdots,\ \ i = 1,2$). Thus their analysis mainly concerns flows of the same nonlinearity. Also there is some difference of interest between the two works. Our main interest is the possibility of precise phase locking between flows with completely different nonlinear parameters, and the possibility of formation of phase-locking states with new nonlinearity from them depending on the coupling weight. On the other hand, the main interest in [4] is the possibility of the phase synchronization in an extended sense, rather than the precise phase locking, when flows of identical nonlinearities are coupled together with equal weight. They call it ''phase synchronization'' if the phase difference is varying but does not grow, that is, when there is no difference in the average angular velocities over long time. They found an interesting threshold in the coupling strength depending on the difference of angular velocities; beyond it the phase difference does not grow over 2π and below it the phase difference grows indefinitely. They also showed that two different flows may phase synchronize in this weak sense when the difference of the two flows may be regarded as a large effective noise term as in the case of the coupling of Rössler and Mackey-Glass systems.

Our phase synchronization (the metamorphosis and the phase locking) may be regarded as an interesting case of the dynamics reduction in the complex system. The global coupling is not essential; even with the nearest-neighbor both-way coupling we observe similar phase synchronization over the network. Most amazing is the fact that even the flows with completely distinct nonlinear parameters can phase synchronize. In this article we clarify the mechanism of this reduction of the network dynamics and investigate how the flows mutually decide their final nonlinearity. In Sec. II we recapitulate some of our previous results. We in particular discuss the condition for the smooth flow limit of the evolution of our network model. In Sec. III we present an intriguing observation that the final phase-synchronizing attractors represent essentially the motion of the ''center of mass'' of the network. The relative motions are swept away by the overwhelming coherence over the network as a result of the synchronization among the flows, which is reminiscent of the appearance of the order parameter in the superconductor or the Higgs condensation in particle physics. Most interestingly the nonlinearity of the final phase-synchronizing attractors is decided mutually among the flows by a simple decision rule so that the final phase-synchronizing attractors may be controlled by the coupling parameters between the flows and/or by the population ratios between unlike flows. In Sec. IV we conclude with some remarks on the flow-map correspondence.

II. THE PHASE SYNCHRONIZATION

First let us present a simple network of globally coupled nonlinear flows. Our model is a natural extension of the globally coupled one-dimensional map lattice by Kaneko [2] to the network of flows and to the higher dimensions. As a canonical example of nonlinear flow with more than one variable, let us take N Lorenz flows and set the parameters for the first set of flows $[(x_i(t), y_i(t), z_i(t)),\ \ i = 1, \ldots, N_1]$

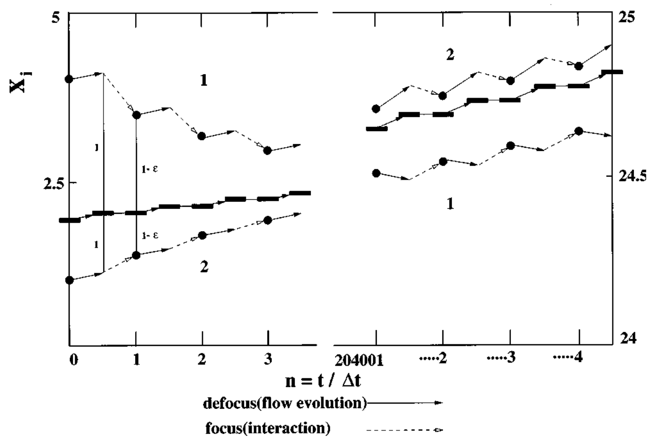

FIG. 1. A sample result of the x-coupled two distinct Lorenz flows, $r_1=28$, $r_2=300$, and the coupling $\epsilon=0.3$, $\theta=0.3$, illustrating the iteration of the two steps; the evolution (the arrowed solid line) and the interaction (the arrowed dashed line). $b=\frac{8}{3}$, $P=10$ for both flows, and only the directly coupled variables x_1 and x_2 are exhibited. The flows (circles) are pulled to the mean field (solid bar) at the fixed rate $1-\epsilon$ by the interaction while the mean field is not affected. For the first few iterations (left) the interaction violates the smoothness of the flows. After sufficient focus (right) the smoothness is realized. Note the change in the scale. The numbers along the horizontal axis represent the iteration steps; each step takes $\Delta t=10^{-4}$. The interaction is instantaneous, but to illustrate the invariance of the mean field, it is represented with half-width of the Δt.

in the chaotic regime and for the second set $[(x_i(t),y_i(t),z_i(t)),\ i=N_1+1,\ldots,N_1+N_2]$ in the periodic regime. Typically we choose $r^{(1)}=28$, $r^{(2)}=300$, $b^{(1)}=b^{(2)}=\frac{8}{3}$, $P^{(1)}=P^{(2)}=10$. At each time step all flows first evolve independently via the flow equations,

$$x_i(t+\Delta t)=x_i(t)+P(y_i-x_i)\Delta t,$$

$$y_i(t+\Delta t)=y_i(t)+(-x_iz_i+r_ix_i-y_i)\Delta t,$$

$$z_i(t+\Delta t)=z_i(t)+(x_iy_i-bz_i)\Delta t,$$

$$r_i=\begin{cases} r^{(1)} & \text{for } i=1,\ldots,N_1 \\ r^{(2)} & \text{for } i=N_1+1,\ldots,N_1+N_2. \end{cases}$$

$$(i=1,\ldots,N) \tag{1}$$

Then, they interact with each other in only one dimension via their mean field. For instance, choosing x for this dimension, the interaction is given by

$$x_i \leftarrow (1-\epsilon)x_i+\epsilon\overline{x}, \tag{2}$$

where $\overline{x}$ is the mean field,

$$\overline{x}=\frac{1}{N}\sum_{i=1}^{N} x_i, \tag{3}$$

and ϵ is the coupling strength. The network evolves repeating this two-step process of nonlinear evolution (1) and interaction (2). We illustrate in Fig. 1 the iteration of the two steps schematically for the x variables in the case of $N=2$. In the first step the flows evolve independently of each other from different values and with distinct nonlinearity. Thus the nonlinear evolution in the first step acts in general as a *defocusing lens*. In the second step the interaction serves as a *focusing lens* with a fixed rate $1-\epsilon$ to the mean field $\overline{x}$. A few remarks are in order here. First, the interaction in the second step preserves the value of the mean field,

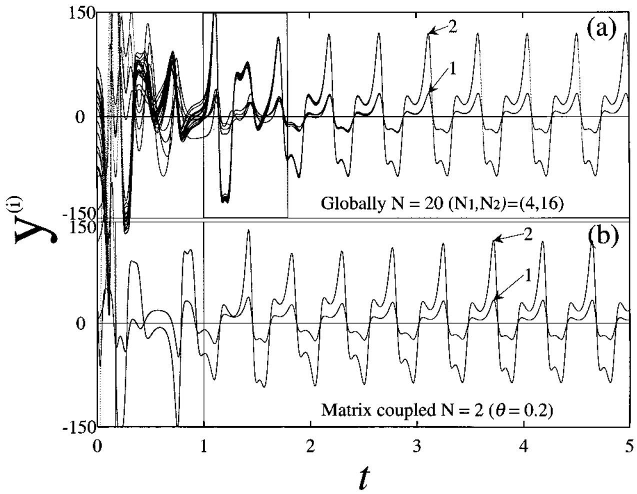

FIG. 2. The evolution of x-coupled Lorenz flows in time: (a) Globally coupled flows; $N_1=4$ flows with $r=28$ and $N_2=16$ flows with $r=300$, both with $b=\frac{8}{3}$, $P=10$. Flow 2 is scaled up by factor 3. (b) Matrix-coupled two flows [see Eq. (9)] at $\epsilon=0.3$ and $\theta=0.2=[4/(4+16)]$. Both from random start. The flows in (a) first form two clusters and passing the decision process (shadowed area) they metamorphose into perfectly phase-synchronizing periodic attractors similar to the final attractors of the matrix model in (b).

$$\frac{1}{N}\sum_i\left((1-\epsilon)x_i+\frac{\epsilon}{N}\sum_j x_j\right)=\frac{1}{N}\sum_i x_i=\overline{x}. \tag{4}$$

This invariance of the mean field under the interaction plays an important role in guaranteeing the continuous motion of the center of mass of the network as is discussed below. Secondly, in our two-step evolution model, the mean field determines the next positions of the flows directly rather than via the velocities of the flows. This is a crucial difference between our model and some similar network model [4,5]. Only in our approach can the dynamics of the network of flows have a direct correspondence with the globally coupled map lattice [2]. Thirdly, although the sequence of nonlinear evolution and interaction is similar with that in the coupled map lattice, we couple the variables in only one dimension to create the mean field and the variables in other dimensions are evolving under the influence of this mean field. Hence it is legitimate to regard the mean field $\overline{x(t)}$ as the master system and the other variables as the slave system [5,6].

Now, let us derive a crucial condition for the smoothness of the evolution of flows under Eqs. (1) and (2). For some period the interaction in the second step pulls the x_i variables to their mean value $\overline{x}$ drastically at every iteration. But, if the focusing is operated on them sufficiently frequently, all of them are soon focused around the mean value $\overline{x}$ and the change of the x_i becomes compatible with or smaller than

Δt. Then the invariance of $\overline{x}$ under the interaction guarantees the smoothness of the evolution of x_i variables to the same extent with the smoothness of other slave variables. Let us denote the time between two Poincaré shots as T. The focusing rate during this period may be estimated as

$$(1-\epsilon)^{T/\Delta t}\approx e^{-\epsilon T/\Delta t}. \tag{5}$$

This should overcome the defocusing rate by the nonlinearity, which is exp ($\lambda_{\max}T$) where $\lambda_{\max}$ is the largest eigenvalue of the Floquet matrix. Thus the condition that focusing works on the x_i variables sufficiently may be estimated

$$\epsilon\frac{T}{\Delta t}>\lambda_{\max}T. \tag{6}$$

After this smooth flow limit is reached we can study the response of the slave system y_i and z_i under the influence of the master system $\overline{x}$. For our case of the Lorenz flows $\lambda_{\max}$ is of the order of 1 and study the region $0.001\leqslant\epsilon\leqslant 1$, so we need $\Delta t\leqslant 10^{-3}$. Of course such Δt is small enough so that the difference equation (1) approximates the differential equation. With this consideration we take $\Delta t=10^{-4}$ throughout this article.

In Fig. 2(a) we show the evolution of the globally coupled $N=20$ flows from random start where $N_1=4$ (chaotic) and $N_2=16$ (periodic). For the first period ($0.2\leqslant t\leqslant 1$) the like flows synchronize among themselves, forming two clusters and, passing the decision period ($1\leqslant t\leqslant 1.8$), the two clusters metamorphose into two final attractors. Let us verify that the first synchronization ($t\leqslant 1$) among like flows is sufficiently robust so that the N flows really pass the decision stage as tightly bound two clusters. Such a test of the dynamics reduction from N to 2 may be devised by constructing a model of two flows representing the two clusters, respectively. If such reduction really occurs, that is, if

$$(x_i,y_i,z_i)=\begin{cases}(x^{(1)},y^{(1)},z^{(1)}) & \text{for } i=1,\ldots,N_1\\ (x^{(2)},y^{(2)},z^{(2)}) & \text{for } i=N_1+1,\ldots,N_1+N_2\end{cases}$$

holds all the time, the evolution equation reduces to

$$x^{(i)}(t+\Delta t)=x^{(i)}(t)+P(y^{(i)}-x^{(i)})\Delta t,$$

$$y^{(i)}(t+\Delta t)=y^{(i)}(t)+(-x^{(i)}z^{(i)}+r^{(i)}x^{(i)}-y^{(i)})\Delta t,$$

$$z^{(i)}(t+\Delta t)=z^{(i)}(t)+(x^{(i)}y^{(i)}-bz^{(i)})\Delta t$$

$$(i=1,2)$$

and the interaction is simply

$$x^{(i)}\leftarrow(1-\epsilon)x^{(i)}+\epsilon\overline{x}\quad(i=1,2), \tag{7}$$

where $\overline{x}$ is the population-ratio-weighted average,

$$\overline{x}=\eta x^{(1)}+(1-\eta)x^{(2)}, \tag{8}$$

with $\eta=N_1/N$ and $1-\eta=N_2/N$. Now let us introduce an $N=2$ flow model with the evolution as given by Eq. (7) and with an interaction in only one dimension,

$$x^{(1)}\leftarrow(1-\epsilon_2)x^{(1)}+\epsilon_2 x^{(2)},$$
$$x^{(2)}\leftarrow(1-\epsilon_1)x^{(2)}+\epsilon_1 x^{(1)}, \tag{9}$$

with $\epsilon_1=\theta\epsilon,\epsilon_2=(1-\theta)\epsilon$. This is a natural extension of the one-way coupling model by Pecora and Carroll [6] to both-way coupling with an interpolation parameter θ. Now we have the interaction described by Eqs. (7) and (8) with the population ratio η on one hand, and the interaction (9) with the interpolation parameter η on the other hand. By simple arithmetic we can show these agree with each other when $\theta=\eta$ [3]. That is, if the reduction from N to 2 really occurs, the evolution of the clusters thereafter, in particular the metamorphosis (decision), should proceed just in the same way with the evolution of the simple $N=2$ matrix model with the interpolation parameter θ set at the value of the population ratio η of the clusters.

In Fig. 2(b) we show the evolution of the matrix-coupled $N=2$ model. The agreement between Figs. 2(a) and 2(b) is remarkable. The flows of the global network in Fig. 2(a) take some time for the formation of two clusters ($0.2\leqslant t\leqslant 1$) and their evolution thereafter into the final attractors is just the same with that of the two flows in Fig. 2(b), with θ adjusted at the population ratio $\eta(=\frac{4}{20}=0.2)$ of the $N=20$ flows in Fig. 2(a).

Till now we have for simplicity divided the flows into two groups, one in the chaotic and the other in the periodic regime. Everything goes the same way with the other combinations, chaotic and chaotic but with different chaoticness and so on. Furthermore we have checked the case of three groups, four groups, . . . , N groups (N flows each with its own nonlinearity). The flows again first synchronize among like flows forming clusters (except for the last case), and then metamorphose into final phase-synchronizing three, four, . . . ,N attractors. Now let us proceed to the target in this article, namely, the analysis of the nonlinearity decision in the metamorphosis.

III. THE METAMORPHOSIS AND NONLINEARITY DECISION

Since we have verified the reduction of the dynamics of the flows to the dynamics of clusters, we hereafter mainly consider the nonlinearity decision between two flows (two clusters) with distinct nonlinearity. This is the study of the second reduction from $N=2$ to 1. (Every discussion below, in particular the extraction of the center of mass degree of freedom, can be extended to the case of general N. Recall the classical mechanics where the two body problem has all the essential ingredients of the N body problem apart from the more subtle case of integrability.) Previously we reported that the population ratio $\eta=N_1/N$ in the globally coupled network of flows, or equivalently the interpolation parameter θ in the matrix-coupled two flows, serves as a control parameter of the nonlinearity of the final attractors. Now we show that there exists a simple rule of the nonlinearity decision which is valid for a wide class of nonlinear flows.

In the limit $\theta=0$ or 1 our matrix model reduces to precisely the master-slave model proposed by Pecora and Carroll in their pioneering work of the chaos synchronization [6]. (In preparing this article we find that these authors also suggested in another paper [7] the study of ''large parameter variation'' which is precisely the target of our work.)

At $\theta=1$ the flow (cluster) 1 wins and at $\theta=0$ the flow (cluster) 2 wins the decision game [3,6]. Thus the rule for $\theta\approx 0$ and $\theta\approx 1$ may be schematically expressed

$$F(r^{(1)})\otimes F(r^{(2)})|_{\theta}\rightarrow\begin{cases}\mathbf{A}(r^{(1)}) & \text{if } \theta\approx 1\\ \mathbf{A}(r^{(2)}) & \text{if } \theta\approx 0,\end{cases}\qquad(10)$$

with obvious symbols. For instance, $\mathbf{A}(r^{(1)})$ means two phase-synchronizing periodic attractors with nonlinearity at $r^{(1)}$. Surprisingly, we find that by adjusting the coupling θ appropriately, it is possible to create from two chaotic flows the phase-synchronizing two periodic attractors

$$F(C,r^{(1)})\otimes F(C,r^{(2)})\rightarrow\mathbf{A}(P)$$

or even to create from the coupling of the two periodic flows the phase-synchronizing two chaotic attractors

$$F(P,r^{(1)})\otimes F(P,r^{(2)})\rightarrow\mathbf{A}(C).$$

Here we added the symbol C or P as a memento of the nonlinearity (chaotic or periodic) of the single flow (cluster) F at the value of $r^{(i)}$ written to the right of it. We assert that the nonlinearity decision rule in general is

$$F(r^{(1)})\otimes F(r^{(2)})|_{\theta}\rightarrow\mathbf{A}(r(r^{(1)},r^{(2)},\theta)),$$

$$r(r^{(1)},r^{(2)},\theta)\approx\overline{r}\equiv\theta r^{(1)}+(1-\theta)r^{(2)}.\qquad(11)$$

The rule is as follows. First, the final two phase-synchronizing attractors are modulo a scale factor the same with the attractor of the original single flow with new nonlinearity. Second, the decided nonlinearity parameter value $r(r^{(1)},r^{(2)},\theta)$ is essentially given by the weighted average $\overline{r}$. (For the network of the flows with two groups of distinct nonlinearity it suffices to replace θ by the population η. The decision rule holds even for the negative θ or for $\theta\geqslant 1$ though for the η such an extension is immaterial.) This means that by varying the parameter θ the whole pattern of the attractor of the single flow in the range $r\in[r^{(1)},r^{(2)}]$ can be produced. The decision rule (11) is so simple that things might look trivial. However, we should note that the fact that the two flows (two clusters) with completely distinct nonlinearity synchronize in phase is already a surprise and that averaging or interpolating the nonlinearity parameters is a completely new notion.

Let us now present evidence for the decision rule. In Fig. 3 we show the Poincaré map of the coupled Lorenz flows for the whole range of the coupling θ with both flows set in the chaotic regime ($r^{(1)}=28$ and $r^{(2)}=200$). Figures 3(a) and 3(b) are for the x-coupled case and for the y-coupled case, respectively, and these should be compared with the Poincaré map of a single Lorenz flow in Fig. 3(c) for $r\in[28,200]$. In Figs. 3(a) and 3(b) the clouds of lower (upper) points represent the Poincaré section for the flow 1 (2). At any value of θ the lower and upper clouds agree with each other after a scaling as the result of the phase synchronization. Around $\theta=1(0)$ the master flow is the system 1 (2) and around these regions the Poincaré map of the final attractors certainly agrees with the Poincaré map of the master flow in Fig. 3(c) as the natural consequence of the rule (10). More interestingly we find in Fig. 3(a) an outstanding periodic window around $0.2\leqslant\theta\leqslant 0.3$ and also many other smaller windows. This shows that two chaotic flows have metamorphosed into phase-synchronizing periodic attractors at certain values of θ. The spectrum of the periodic windows in θ in Fig. 3(a) precisely agrees with the spectrum of the windows of the single flow in $\overline{r}$ in Fig. 3(c). Furthermore the z distributions of the clouds in Fig. 3(a) at any θ agree with the z distributions in Fig. 3(c) at the corresponding value of $\overline{r}$, which means that the final phase-synchronizing attractors are nothing but the attractor of the single flow at the corresponding $\overline{r}$. Summing up we observe that in the case of the x coupling the extended decision rule (11) holds precisely in the form

$$F(C,r^{(1)}=28)\otimes F(C,r^{(2)}=200)|_{\theta}$$

$$\rightarrow\mathbf{A}(\overline{r}=28\theta+200(1-\theta)).\qquad(12)$$

Notably the spectrum of the periodic windows in Fig. 3(b) is almost the same as that in Fig. 3(c) but the positions of the windows in θ are shifted to the smaller θ direction about 0.15. This on one hand reveals that the decision rule holds in general approximately and on the other hand indicates the nontriviality of the rule. The agreement in the Lyapunov exponents between Figs. 3(d), 3(e) (coupled flows), and 3(f) (a single flow) also confirms the extended decision rule.

Now let us clarify why the decision rule holds. We divide the argument into items.

(1) *Decomposition of the variables*. When the final phase-synchronizing attractors are formed after the metamorphosis, the variables $x^{(1)}$ and $x^{(2)}$ have already focused around the mean value $\overline{x}$ while the other variables $y^{(1)},z^{(1)},y^{(2)}$, and $z^{(2)}$ evolve nonautonomously under the influence of $\overline{x}$. In order to analyze their motion let us introduce the ''center of mass'' variables and the relative variables just as in the classical mechanics. That is,

$$\overline{(x,y,z)}=\theta(x^{(1)},y^{(1)},z^{(1)})+(1-\theta)(x^{(2)},y^{(2)},z^{(2)}),$$

$$(x_R,y_R,z_R)=(x^{(1)},y^{(1)},z^{(1)})-(x^{(2)},y^{(2)},z^{(2)}).\qquad(13)$$

Some explanation of the term ''center of mass'' is in order here. We have defined $\overline{x}$ in Eq. (3) for the network of flows which leads to $\overline{x}$ in Eq. (8) for the two clusters (and the $\overline{x}$ above via the identification $\theta=\eta$). These $\overline{x}$ might have been called the x coordinate of the center of mass of N flows and that of two clusters, respectively, with an assignment of a unit mass to each flow. However, we carefully called them the mean field and the population-ratio-weighted average, respectively, since they are concerned with only one of the dimensions. Here we are newly defining $\overline{y}$ and $\overline{z}$ in accord with $\overline{x}$ to define a vector $\overline{(x,y,z)}$. Hence we may now call this the center of mass. We admit that the decomposition above is really a simple algebraic redefinition of the variables but we use the term *center of mass* hereafter in order to emphasize the conceptual jump that we consider not only the active variable $\overline{x}$ (active in the sense that it is used in the model to impose the focusing on the flows) but also fictitious $\overline{y}$ and $\overline{z}$ together. We are aware that it is rather radical to talk

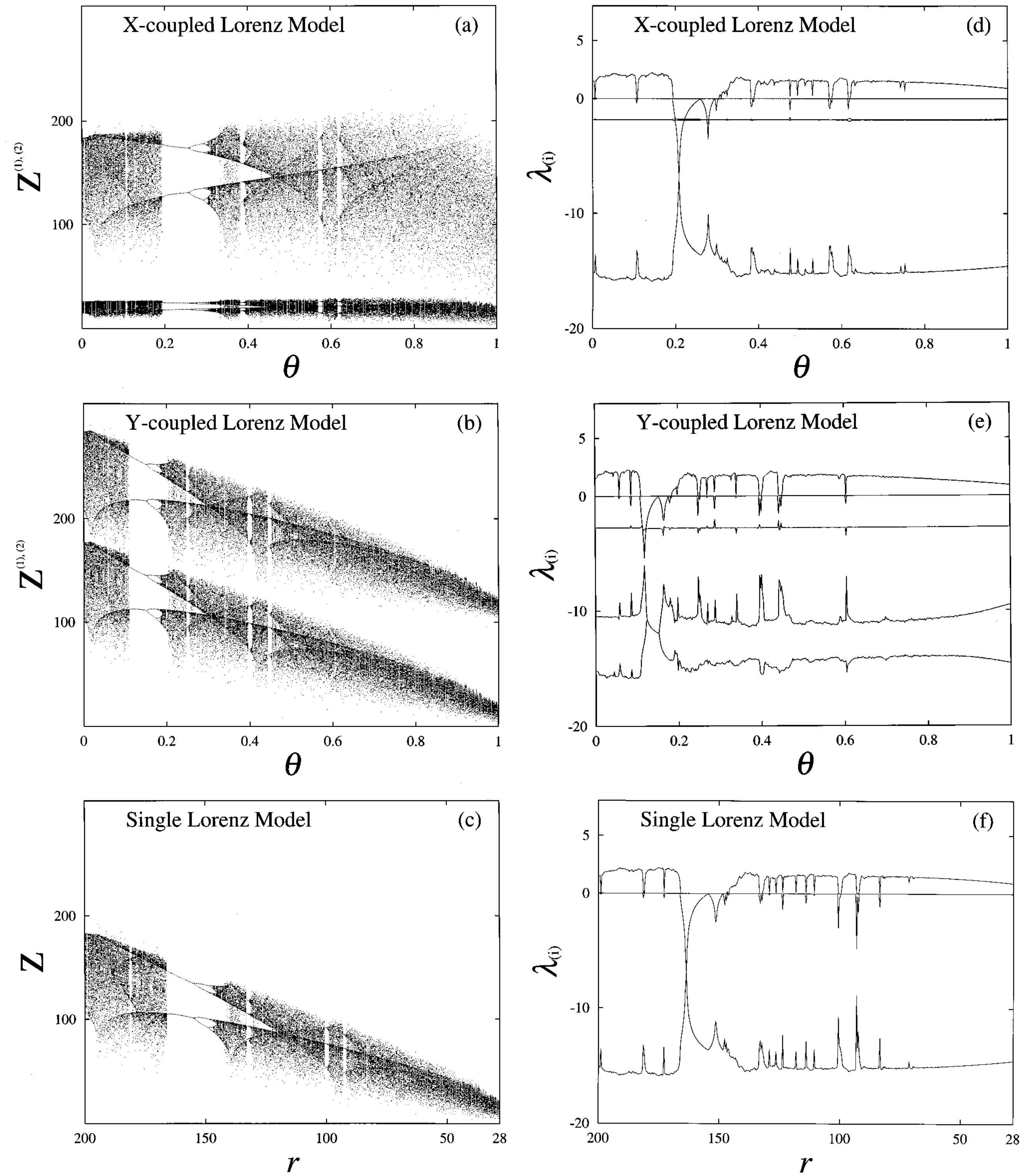

FIG. 3. The Poincaré map of two chaotic Lorenz flows ($r_1=28$, $r_2=200$, $b=\frac{8}{3}$, $P=10$) coupled by a matrix with $\epsilon=0.3$ and $\theta \in [0,1]$ [see Eq. (9)], (a) x coupling, (b) y coupling, to be compared with the Poincaré map of the single flow $r\in[200,28]$) in (c). (For the y coupling the map of flow 2 is shifted upwards to avoid the overlap.) The maps are sampled by the conditions $dz/dt=0$ and $d^2z/dt^2=0$. The periodic window spectrum of (a) [(b)] agrees perfectly (approximately) with that of (c). The corresponding Lyapunov exponents in (d), (e), and (f) confirm the observation. See the decision rule (12).

about the center of the mass of the attractors but this analogy actually turns out vital in the discussion below. We also decompose $r^{(1)}$ and $r^{(2)}$ as

$$\overline{r}=\theta r^{(1)}+(1-\theta)r^{(2)},$$

$$\Delta r=r^{(1)}-r^{(2)}.$$

(2) *The motion of the center of mass.* With these new variables we can rewrite Eq. (1) as

$$\overline{x} \leftarrow \overline{x} + P(\overline{y} - \overline{x})\Delta t,$$

$$\overline{y} \leftarrow \overline{y} + (-\overline{x\ z} + \overline{r\ x})\Delta t + \theta(1-\theta)(-x_R z_R + x_R \Delta r)\Delta t, \tag{14}$$

$$\overline{z} \leftarrow \overline{z} + (\overline{x\,y} - b\overline{z})\Delta t + \theta(1-\theta) x_R y_R \Delta t.$$

The flows (clusters) evolve under the iteration of the two-step process of the evolution and interaction. For the x coupling the x_R becomes quickly of the order of Δt in the iteration. Taking $x_R \rightarrow 0$ in Eq. (14) we find that the terms proportional to $\theta(1-\theta)$ all vanish and the evolution of the center of the mass $(\overline{x,y,z})$ in the first step reduces to the evolution of the single Lorenz flow with the nonlinearity parameter $\overline{r}$. As for the second step the invariance condition (4) comes into play. It assures that the center of the mass $(\overline{x,y,z})$ is not affected by the interaction. Hence the $(\overline{x,y,z})$ evolves only by just the first step in each iteration. Thus the center of the mass of the two clusters and the center of mass of the N flows should form the Lorenz attractor with nonlinearity $\overline{r}$.

(3) *The motion of phase-synchronizing orbits and the decision rule*. The final two attractors $(x^{(1)},y^{(1)},z^{(1)})$ and $(x^{(2)},y^{(2)},z^{(2)})$ are phase synchronizing with each other. Hence their orbits must agree with the orbits of their center of mass modulo a scale factor. Hence the final attractors must be those of the single Lorenz flow with nonlinearity $\overline{r}$. This is the explanation of the precise decision rule for the x-coupling case.

(4) *The reduction from $N=2$ to 1*. The original degree of freedom of two flows (clusters) was 6, and after $x_R \rightarrow 0$ it becomes 5, the $\overline{x,y,z}$, and the rotation and expansion around the origin in the $x_R - y_R$ plane.

The phase synchronization, namely, the synchronizing motion around similar orbits, implies that the last two freedoms are essentially also lost. This is clearly seen in the Lyapunov exponents in Figs. 3(d) and 3(e). There are six eigenvalues among which three precisely agree with the exponents of the single Lorenz flow in Fig. 3(f) in the whole range of $\overline{r}$. They change sensitively with the variation of the coupling θ and represent the active degree of freedom of $(\overline{x,y,z})$. The other exponents do not vary with θ and represent the nonactive degree of freedom. From the nonautonomous evolution equations for (x_R, y_R, z_R) we can easily verify the independence of their Lyapunov exponents from θ.

(5) *Generic cases*. In the above argument we used the fact that x_R is focused to nearly zero in the x coupling. This removes the terms proportional to $\theta(1-\theta)$ and reduces Eq. (14) to the evolution equation of the single Lorenz flow at $\overline{r}$. The sufficient condition under which the same argument works in the generic case is that the evolution equation does not have nonlinear terms in other variables than the one chosen for the coupling. In short the slave dimensions must be linear among themselves. In the case of the above x-coupled Lorenz system, the nonlinear terms are xz and xy, both of which are linear in y and z. Hence the sufficient condition is satisfied and the decision rule holds precisely. Actually the synchronization of the coupled flows can be realized in general when the Lyapunov exponents for the driven system are less than zero [6]. For the Lorenz flows we can construct the coupled flow network not only by the coupling in x but also in y. For the y-coupled Lorenz flows, the xz term does not satisfy the sufficient condition. However, as we find in the numerical calculation, the nonlinearity decision is approximately made following the rule in Eq. (11) with some negotiation (the quoted shift by ~ 0.15). This can be understood as due to the low nonlinearity in the slave dimensions. We have numerically tested various known nonlinear flows. In all cases the decision rule holds precisely when the sufficient condition is satisfied, and approximately otherwise so far as the synchronization occurs. For instance, the Rössler flows can be synchronized by the y coupling. The nonlinear term xz violates the sufficient condition and we observe the negotiated decision (see Fig. 4 below). The Brusselator can be synchronized by both x and y coupling. The nonlinear term x^2y satisfies the condition for the x coupling but violates it for the y coupling. We indeed observe that the rule holds precisely in the x coupling but approximately in the y coupling.

Let us present two more figures in order to demonstrate how the decision rule works and to indicate the feasibility of the technical application of it. In Fig. 4 we choose the y-coupled Rössler model which exhibits the negotiated decision. The bottom box represents the Lyapunov exponents in the range $c \in [2,6.5]$. The other parameters a and b are both fixed at 0.2. We pick two flows, the flow 1 with $c=3.5$ and the flow 2 with $c=5.3$.

Before the coupling both flows are in the periodic regime as shown in the middle two boxes. By the coupling with $\epsilon=0.3$ and $\theta=0.2$ they should metamorphose following the rule:

$$\mathcal{F}_{\text{Rössler}}(P,c^{(1)}=3.5) \otimes \mathcal{F}_{\text{Rössler}}(P,c^{(2)}=5.3)|_{\theta=0.2} \rightarrow \mathbf{A}(c), \tag{15}$$

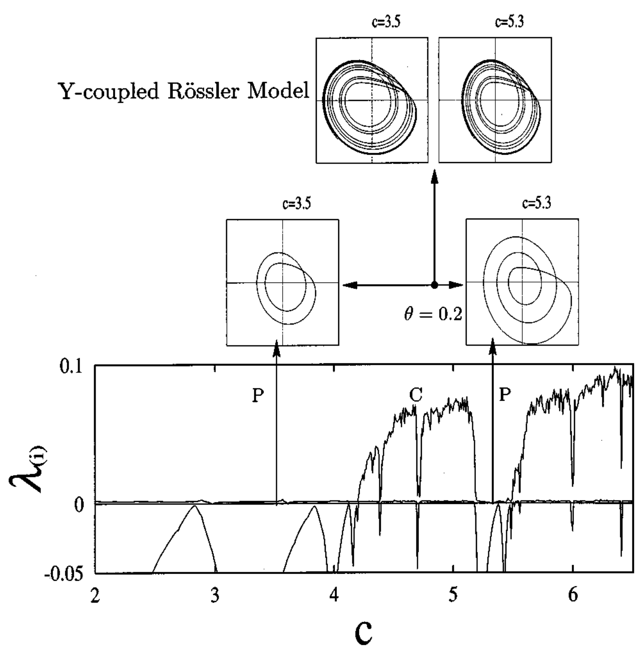

FIG. 4. An illustration of the decision rule in Eq. (15). We pick two Rössler flows $c_1=3.5, c_2=5.3, a=b=0.2$; the attractors are both periodic (middle two boxes). The coupling with $\epsilon=0.3$ and $\theta=0.2$ makes chaotic phase-synchronizing two attractors (top two boxes).

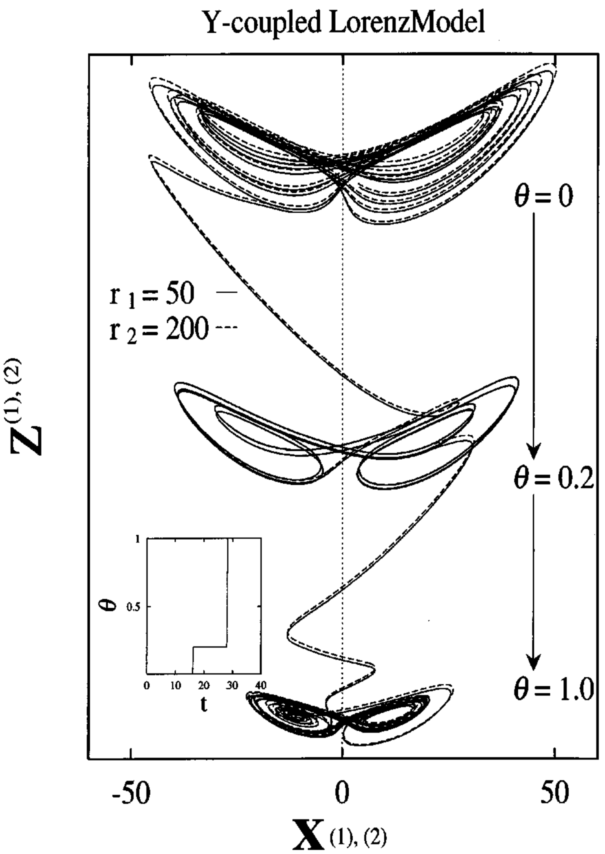

FIG. 5. A sample of control of phase-synchronizing two coupled attractors (solid and dashed lines, $\epsilon=0.3$) by the coupling θ. Flows 1 and 2 are both chaotic. $r_1=50$, $r_2=200$, $b=\frac{8}{3}$, $P=10$. Responding to the quick change of the coupling depicted in the little box they swiftly decide their mutual nonlinearity by the decision rule (16). At the rest period $t\in[16.2,28]$ they freely make periodic attractors (the middle profile).

where $\mathcal{F}_{\text{Rössler}}$ represents the Rössler flow, and $c\approx\overline{c}=3.5\times0.2+5.3\times0.8=4.94$. The single flow at $c\approx4.94$ is chaotic, as can be seen in the bottom box with the maximum Lyapunov exponent $\lambda_{\max}=0.0693$. The final phase-synchronizing attractors shown in the top two boxes are indeed chaotic with $\lambda_{\max}=0.071$ and the decision rule is respected by two Rössler flows well even in the y coupling.

In Fig. 5 we exhibit the phase-synchronizing attractors of the y-coupled Lorenz flows ($\epsilon=0.3$) controlled by the decision rule

$$\mathcal{F}_{\text{Lorenz}}(C,r^{(1)}=50)\otimes\mathcal{F}_{\text{Lorenz}}(C,r^{(2)}=200)|_{\theta}\rightarrow\boldsymbol{A}(r(r^{(1)},r^{(2)},\theta)),$$

$$r(r^{(1)},r^{(2)},\theta)\approx\overline{r}\equiv\theta r^{(1)}+(1-\theta)r^{(2)},\tag{16}$$

where $\mathcal{F}_{\text{Lorenz}}$ represents the Lorenz flow. [This supplements our previous analysis of the simpler case $\mathcal{F}_{\text{Lorenz}}(C,r=28)\otimes\mathcal{F}_{\text{Lorenz}}(P,r=300)|_{\theta}$. See Fig. 2 of [3].] Actually this figure is a record of controlling the phase-synchronizing attractors by varying θ by pressing the key of a personal computer. In order to avoid the overlap of the trajectories the attractors are constantly scrolled down in the display. Most intriguing is the robustness of the phase synchronization under the rapid variation of the coupling θ. See the change of θ during the period $t\in[16,16.2]$ and $t\in[28,28.2]$ reproduced in the little box. We tried a game to produce the periodic attractors from two chaotic flows by control. Watching the two phase-synchronizing attractors dancing in the display it was quite easy to figure out the value of θ necessary, which was around 0.2. Thus in the run for this figure we let the two flows move around freely with θ fixed at 0.2 for the period $t\in[16.2,28]$. The orbits in the middle of the figure are the resulting periodic attractors.

The single Lorenz flow has a prominent periodic window around $r=160$. The decision rule gives $\overline{r}=50\theta+200(1-\theta)\approx170$ for $\theta=0.2$ and we have certainly caught this window in the control. How much is the decision rule negotiated by the y-coupled Lorenz flows? The Lyapunov exponents of these attractors are $(\underline{0},-\underline{1.46},-2.93,-8.69,-\underline{13.6},-326)$ and the underlined three exponents agree with three exponents of the single Lorenz flow at $r=165.6$. While the precise rule under the sufficient condition dictates the formation of the flows with $r=170$ the actual attractors have $r=165.6$. Thus the negotiation is $170\rightarrow165.6$.

IV. CONCLUSION

The dynamics of a network of globally coupled flows with distinct nonlinearity is reduced first into that of clusters formed by like flows by synchronization among them and then further reduction occurs by the decision of the nonlinearity among the clusters. The clusters metamorphose into the final phase-synchronizing attractors which are essentially the attractor of a single flow with mutually decided nonlinearity among the clusters. We have shown that the decision rule can be written in a simple form in Eq. (11). The sufficient condition for the decision rule is that the nonlinear terms of the flow equation are linear in the variables other than the one chosen for the coupling. Even if the condition is not satisfied the decision rule is respected with some negotiation as long as the synchronization can be realized between the flows (clusters). We have shown ample examples for both cases and have demonstrated the feasibility of technical application of the rule.

We close this article by pointing out that our network model of globally coupled flows may be closely linked to the network of globally coupled maps. Recall the intriguing fact that the flow of a dissipative system may be in the same universality class with the map essentially because the Poincarié section of the flow becomes one dimensional due to dissipation [8]. We have used the condition (6) to determine Δt so that the focusing (2) is applied to the system frequently enough and that the smooth flow limit is guaranteed after Δx becomes very small. Thus the analysis presented in this article is the dynamics in the strong focus regime where the dynamical reduction overwhelms the network. If, on the other hand, ϵ is extremely small, as small as $O(\Delta t)$, the effect of interaction will not affect the smoothness of the orbits, and the nonlinearity of the flows and the coherence due to the interaction via the mean field will make a subtle balance on the Poincaré section. In such a weak focus regime, the network of flows will mimic the network of maps on the Poincaré section. In a preliminary analysis of the coupled Duffing oscillators in the universality class of the May map, we indeed observed the formation of spatial clusters. An extensive study in this regime is underway.

[1] E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990).
[2] K. Kaneko, Phys. Rev. Lett. **63**, 219 (1989); Physica (Amsterdam) **41D**, 137 (1990).
[3] H. Fujigaki, M. Nishi, and T. Shimada, Phys. Rev. E **53**, 3192 (1996).
[4] M. G. Rosenblum, A. S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **76**, 1804 (1996); A. S. Pikovsky, M. G. Rosenblum, and J. Kurths, Europhys. Lett. **34**, 165 (1996).
[5] J. M. Kowalski and G. L. Albert, Phys. Rev. A **42**, 6260 (1990).
[6] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).
[7] T. L. Carroll and L. M. Pecora, Physica **67**, 126 (1993).
[8] M. J. Feigenbaum, Los Alamos Sci. **1**, 4 (1980); in *Universality in Chaos*, edited by P. Cvitanović (Hilger, London, 1989).